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1993 J. Phys. A: Math. Gen. 26 3085

(http://iopscience.iop.org/0305-4470/26/13/013)

J. Phys. A: Math. Gen. 26 (1993) 3085–3091. Printed in the UK

Directed animals on two-dimensional lattices

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Received 9 October 1992

Abstract. Previously, directed animals on square and triangular lattices have been enumerated by area, and have been found to have simple generating functions, whilst the hexagonal lattice generating function has not been obtained. In this paper, directed animals on several new lattices are enumerated, one class of which is solved exactly. Directed animals by bonds (with and without loops) are also enumerated. In each case an asymptotic growth like $n^{-1/2}\mu^n$ is observed and precise estimates for μ are given.

1. Introduction

A directed lattice animal $\mathcal{A}$ is a set of points on a lattice such that all points $p \in \mathcal{A}$ are either the (unique) origin point or one (lattice) step in one of the preferred directions from some other point in $\mathcal{A}$. This means that they start at a given origin, and then move out in some preferred directions. Directed animals have relevance to percolation.

Directed animals in two dimensions have been extensively studied [1, 2] on strips of finite width, and have been solved for the square and triangular lattice empirically by Dhar *et al* [3]. They have been shown to be equivalent to the Baxter hard-square problem by Dhar [4], a result subsequently proved for the square lattice by Gouyou-Beauchamps and Viennot [5] through a bijection to one-dimensional walks, and later by Bétréma and Penaud [6] through a bijection to asymmetric trees. The triangular case has also been solved by Viennot [7]. A more general summary is provided in [8] and [9].

We have followed Dhar *et al* [3] by enumerating (by computer) the number of animals of a particular lattice topology, and have then analysed this series using differential [10] and algebraic approximants [11].

In particular, the search was driven by the the fact that square and triangular site-directed animals had a very simple algebraic generating function, whereas hexagonal site-directed animals seemed to lack this property, as pointed out by Dhar *et al* [3].

It was shown by Dhar *et al* [3] that the triangular, square and hexagonal site-directed series grew asymptotically like $n^{-1/2}\mu^n$ where μ is 4 for the triangular lattice, μ is 3 for the square lattice, and μ is a non-integer, 2.0252 ± 0.0005 for hexagonal animals.

We have extended the hexagonal site series from 48 terms to 99 terms without finding an algebraic generating function. Since the square and triangular lattices were found with only a few terms, we conjecture that the hexagonal site-directed animal generating function is not algebraic.

After this surprise, we attempted to find exact algebraic equations for some other lattices (some non-regular). The hexagonal lattice can be fitted onto a square lattice by leaving out some possible connections, producing a 'brickwork' effect. Other lattice types can be impressed upon the square lattice, some of which are not 'normal' lattices. An algebraic generating function was discovered for one set of these lattices. It will be described later.

0305-4470/93/133085+07$07.50

Also, only *site*-directed animals had previously been solved. Two other types of directed animals, bond-directed and bond-directed without loops, were also studied. For no such animals did we find algebraic generating functions. The difference between the three types of animals lies in the way they are counted. An example on the square lattice is the animal consisting of four sites arranged in a square. In site-directed animals, this represents one animal. For bond-directed animals, this represents three animals: two with three bonds (a U and a C shape), and one with four bonds (a full loop). For bond-directed (no loops) animals, the four-bond loop is not allowed, so the four occupied sites only represent the two animals with three bonds.

It is clear that there are at least as many animals counted by bonds as there are with bonds (no loops), and there are at least as many counted by bonds (no loops) as by sites. Accordingly, we expect $\mu(\text{site}) \leqslant \mu(\text{bond, no loops}) \leqslant \mu(\text{bond})$. Empirically, we find that strict inequality holds for the nine lattices considered here.

The family of lattices studied is shown in figure 1.

2. Enumeration method

Direct enumeration was not practical, due to the large size of the numbers produced. Instead, a method similar to that described by Dhar *et al* [3] involving a transfer-matrix-type approach across a diagonal perpendicular to the preferred direction, and making use of mirror symmetries where possible, was developed.

In particular, a computer program was produced which worked on a matrix with a given number of *states*. Each state represented a different variety of site. For instance, in the square lattice, all sites are identical, so there is only one state. In the hexagonal lattice, there are two types of sites: those that can move horizontally; and those that cannot. Thus there are two states for the hexagonal lattice.

The boundary conditions on one diagonal can be expressed as a linear combination of the boundary conditions on the diagonal one unit outwards, after adding a few elements. By storing previously calculated boundary conditions, these can be calculated efficiently.

Note that animals on lattices which include diagonal bonds take longer to enumerate as the size of the boundary is twice as large, and the complexity grows exponentially with the boundary size.

3. Results

A picture of the lattices is given in figure 1. The preferred direction is up and to the right. The series coefficients for lattices with more than one state will depend upon which state is chosen as the initial state, but it is believed that this will not affect the existence or otherwise of an algebraic generating function. In all the instances below, there has been a 'natural' starting state, from which the series has been started. This is the lower-leftmost cell in figure 1.

Three types of animal on nine lattices gave 27 series in all. To save space we do not give the coefficients (in excess of 1000). Instead we give two series, the 99-term hexagonal site animal series (extending Dhar *et al*'s [3] enumeration) in table 1, and the square-lattice bond series in table 2. Other series are available on request from the first named author (email address: arc@mundoe.maths.mu.oz.au).

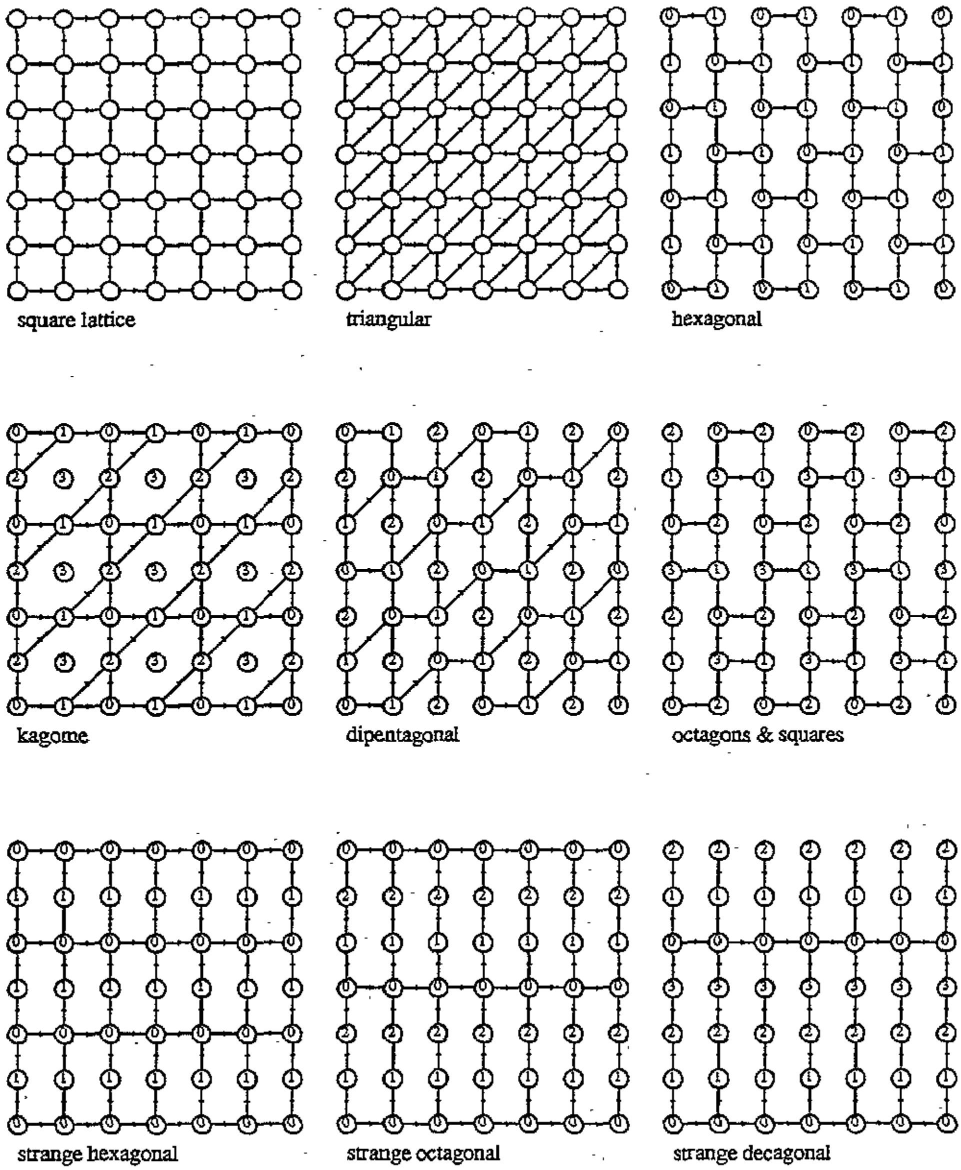

Figure 1. Picture of the nine lattice topologies used.

The analysis consisted of first searching for an exact algebraic equation [11] and, failing that, analysing the series by differential approximants. This method of analysis is now quite standard, and is described in detail in [10].

In table 3 we give a summary of all the results. For each lattice, and for each of the three animal types, we indicate whether it has been solved exactly (for an algebraic generating function), how many terms have been enumerated, and the values of μ obtained

Table 1. Number of animals on the hexagonal lattice.

n	Animals with n sites	n	Animals with n sites
1	1	51	508 363 015 172 300
2	2	52	1 019 972 273 445 851
3	3	53	2 046 808 822 092 474
4	6	54	4 108 071 760 820 439
5	11	55	8 246 469 606 486 634
6	21	56	16 556 365 592 922 836
7	40	57	33 245 018 757 500 920
8	77	58	66 765 272 438 643 476
9	149	59	134 101 946 854 407 712
10	289	60	269 387 847 045 971 641
11	563	61	541 224 581 478 870 387
12	1099	62	1 087 506 471 314 584 006
13	2152	63	2 185 442 159 688 419 859
14	4222	64	4 392 363 747 731 461 439
15	8299	65	8 828 913 289 559 275 069
16	16 339	66	17 748 624 493 635 942 975
17	32 217	67	35 683 647 167 071 197 039
18	63 612	68	71 749 603 806 980 992 331
19	125 753	69	144 282 686 583 360 756 484
20	248 870	70	290 169 750 747 404 964 486
21	493 015	71	583 622 555 534 978 042 575
22	977 576	72	1 173 958 850 272 328 761 795
23	1 940 042	73	2 361 638 724 002 508 525 304
24	3 853 117	74	4 751 303 560 113 863 487 747
25	7 658 211	75	9 559 822 201 935 585 592 685
26	15 231 219	76	19 236 391 212 494 063 209 234
27	30 312 012	77	38 710 894 000 785 915 368 960
28	60 360 046	78	77 907 221 754 039 272 464 621
29	120 260 317	79	156 803 691 779 869 398 257 751
30	239 727 623	80	315 622 580 238 164 271 794 136
31	478 105 086	81	635 348 868 528 451 823 794 284
32	953 950 878	82	1 279 051 682 652 368 025 780 955
33	1 904 209 707	83	2 575 104 252 227 354 120 046 444
34	3 802 587 910	84	5 184 796 405 275 057 586 642 065
35	7 596 437 240	85	10 439 941 640 542 574 227 809 800
36	15 180 921 041	86	21 022 929 514 845 448 173 524 974
37	30 348 394 157	87	42 336 655 543 678 468 248 662 824
38	60 689 739 010	88	85 264 329 601 783 920 697 896 605
39	121 403 119 626	89	171 729 588 933 981 640 090 771 504
40	242 925 445 980	90	345 898 919 606 375 679 391 312 408
41	486 226 668 328	91	696 753 306 957 994 580 743 546 710
42	973 467 761 968	92	1 403 570 007 674 326 674 058 091 786
43	1 949 468 395 563	93	2 827 572 796 439 879 767 639 057 858
44	3 904 970 715 501	94	5 696 625 485 781 401 013 945 751 403
45	7 823 872 468 948	95	11 477 444 474 359 960 449 567 396 803
46	15 679 198 951 587	96	23 125 754 216 512 540 984 858 689 608
47	31 428 242 462 299	97	46 598 218 546 799 030 718 727 870 893
48	63 009 591 480 990	98	93 899 865 311 126 117 655 075 154 741
49	126 351 391 028 540	99	189 226 706 028 309 538 198 211 370 984
50	253 417 639 018 096		

from differential approximant analysis. In all cases the coefficients were found to grow like $\mu^n n^{-1/2}$, so that it seems that bond animals, site animals and loopless bond animals are all in the same universality class.

Table 2. Number of bond animals on the square lattice

n	Animals with n bonds	n	Animals with n bonds
1	1	21	9 075 301 990
2	2	22	31 010 850 632
3	5	23	106 100 239 080
4	14	24	363 428 599 306
5	42	25	1 246 172 974 048
6	130	26	4 277 163 883 744
7	412	27	14 693 260 749 888
8	1326	28	50 516 757 992 258
9	4318	29	173 812 617 499 767
10	14 188	30	598 455 761 148 888
11	46 950	31	2 061 895 016 795 926
12	156 258	32	7 108 299 669 877 836
13	522 523	33	24 519 543 126 693 604
14	1 754 254	34	84 623 480 620 967 174
15	5 909 419	35	292 204 621 065 844 292
16	19 964 450	36	1 009 457 489 428 859 322
17	67 618 388	37	3 488 847 073 597 306 764
18	229 526 054	38	12 063 072 821 044 567 580
19	780 633 253	39	41 725 940 730 851 479 532
20	2 659 600 616	40	144 383 424 404 966 638 976

Note that many other lattices were tried, but were generally uninteresting.

4. Strange lattices

The 'strange' lattices were invented because of their resemblance to the hexagonal lattice. The main reason for our interest in them is that they were found to be solvable (for an algebraic generating function) in the site case, whereas no other lattices tried were solvable (other than those reducible to previously solved cases).

Their generating function $f(x)$ was found to satisfy the equation

$$-x+\left(x^2\frac{1-x^N}{1-x}-3x+1\right)(f+f^2)=0$$

where N is the number of 'gaps' in the lattice unit cell. That is, for the square lattice, $N=0$, for the strange hexagonal lattice $N=1$, for the strange octagonal lattice, $N=2$, etc. This formula has been tested up to $N=4$. Note that this formula was empirically determined by fitting the exact enumeration series to algebraic approximants [11], and has not been proved.

This formula has two obvious limits. First, when $N=0$, the normal square lattice generating function $-x+(-3x+1)(f+f^2)=0$ is recovered, and second as N approaches infinity, a solution of the form $f=x/(1-2x)$ falls out, which is the generating function for animals on infinite vertical strips held together by one horizontal line. It is straightforward to prove that this generating function is correct, as the problem decouples into the vertical bars ($v=1/(1-x)$ each), which with horizontal bonds added gives $f=xv/(1-xv)=x/(1-2x)$.

An unsuccessful but still interesting attempt was made to prove this formula using the method of Bétréma and Penaud [6]. This method constructs an algebraic language which

Table 3. Summary of results for the 27 models studied.

Lattice	Type	Solved	Terms	Connective constant
Square	site	yes	30	3
	bond	no	40	3.507695 ± 10
	no loops	no	40	3.380834 ± 12
Triangular	site	yes	19	4
	bond	no	20	5.6828 ± 2
	no loops	no	20	5.3464 ± 3
Hexagonal	site	no	99	2.025131 ± 5
	bond	no	80	2.177857 ± 5
	no loops	no	80	2.12853 ± 3
Kagomé	site	no	31	2.7010 ± 3
	bond	no	31	3.4274 ± 2
	no loops	no	31	3.26605 ± 10
Dipentagonal	site	no	37	2.81160 ± 2
	bond	no	35	3.07806 ± 10
	no loops	no	35	3.0084 ± 1
Octagons and squares	site	no	123	1.91832 ± 5
	bond	no	100	2.108595 ± 5
	no loops	no	100	2.04794 ± 5
Strange hexagonal	site	yes	42	$2.6180339... = \frac{1}{2}(3+\sqrt{5})$
	bond	no	40	2.7565 ± 1
	no loops	no	40	2.71814 ± 3
Strange octagonal	site	yes	40	$2.4142136\ldots = 1+\sqrt{2}$
	bond	no	40	2.46781 ± 5
	no loops	no	40	2.45203 ± 7
Strange decagonal	site	yes	30	$2.28879499\ldots$
	bond	no	30	2.31286 ± 5
	no loops	no	30	2.30553 ± 6

generates a class of asymmetric trees. Bétréma and Penaud [6] then proved that a bijection exists between this set of trees and the directed lattice animals. Thus enumerating the asymmetric trees is equivalent to enumerating the directed animals. We have extended their grammar for the strange hexagonal lattice:

$$G = \epsilon + r + aG + raG + bbG + xMyyG$$

$$M = E + rE$$

$$E = S + SaM$$

$$S = \epsilon + aSbbS + aSxMyyS.$$

These equations generate a (different) class of asymmetric trees. In particular, G generates this class of trees. The other variables generate subsidiary geometrical objects. If the tree is oriented vertically, then E generates *guingois* trees: trees that never go to the right of the root. S are trees that never go right of the root, and which return to a site directly below the root. M are *guingois* trees that may have a dead right bond. Here r means a 'dead' right bond (only going half way), bb means a 'full' right bond, a means a "left" bond, and the xyy set means a full left and right bond. G is the starting state. The details of the reason for such an approach, and the methods for manipulating it are given in [6] which should be read to understand the above grammar and its motivation.

From the equations of the grammar one obtains the generating function by converting these to algebraic equations, and converting all lower case symbols to x, and ϵ to 1. Then $xG(x) = f(x)$. The factor of x is due to the fact that we count the root site as a site; [6] does not. Although solving this grammar gives the strange lattice generating function, we have *not* been able to find a bijection between the class of trees generated by the language and the directed animals.

Note that this grammar can easily be modified for other values of N as follows. Change the bb to $N+1$ instances of b, change the yy to $N+1$ instances of y, and change the r to $r + r^2 + r^3 + \cdots r^N$. If $N = 0$, the grammar of Bétréma and Penaud [6] results. We conjecture that a simple bijection between these trees and the animals exists, but have yet to find it.

Acknowledgments

We would like to thank Deepak Dhar, whose questions initiated this investigation. Andrew Conway would like to gratefully acknowledge the A O Capell, Wyselaskie and Daniel Curdie scholarships. AJG and RB would like to thank the Australian Research Council for financial support.

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